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LETTER TO THE EDITOR

Lie and Noether symmetry groups of nonlinear equations

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Abstract. We find that the nonlinear ordinary differential equation $\ddot{\rho}+\omega^2(t)\rho=1/\rho^3$ has a three-parameter Lie group of symmetries which are also Noether symmetries. The invariants associated with the group are calculated. We discuss a new way of generating n-parameter Lie symmetry groups which are associated with n-parameter nonlinear differential equations.

Recently Reid and Ray (1982) studied the Lie symmetries of the differential equation

$$\ddot{\rho}+\omega^2(t)\rho-G(t)F(k(t)\rho)=0 \tag{1}$$

where $G(t)$ and $k(t)$ are initially arbitrary functions of time. (Overdots imply time differentiation.) Here we point out that one particular nonlinear equation from the class of equations represented by (1) possesses a three-parameter Lie group of symmetries which are also Noether symmetries. That special equation is the Pinney equation (Pinney 1950)

$$\ddot{\rho}+\omega^2(t)\rho-1/\rho^3=0, \tag{2}$$

for which the symmetry group operator is

$$X=a(t)(\partial/\partial t)+\tfrac{1}{2}\dot{a}\rho(\partial/\partial\rho). \tag{3}$$

The function $a(t)$ in (3) satisfies the linear third-order equation

$$\dddot{a}+4\omega^2\dot{a}+4\omega\dot{\omega}a=0. \tag{4}$$

The general solution to (4) can be written in terms of a particular solution $x_p(t)$ of the auxiliary equation (Eliezer 1979)

$$\ddot{x}+\omega^2(t)x=1/x^3, \tag{5}$$

i.e.

$$a=c_1x_p^2\sin 2\tau+c_2x_p^2\cos 2\tau+c_3x_p^2. \tag{6}$$

The c's are integration constants and τ is a new time variable defined by

$$\tau=\int\frac{dt}{x_p^2}.$$

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By substituting $a(t)$ in (6) into group generator (3) we can identify three elementary generators, one for each of the constants c_1, c_2, c_3. These generators have the forms (Leach 1980)

$$X_1 = x_p^2 \sin 2\tau(\partial/\partial t) + (x_p\dot{x}_p \sin 2\tau + \cos 2\tau)\rho(\partial/\partial\rho), \qquad (7a)$$

$$X_2 = x_p^2 \cos 2\tau(\partial/\partial t) + (x_p\dot{x}_p \cos 2\tau - \sin 2\tau)\rho(\partial/\partial\rho), \qquad (7b)$$

$$X_3 + x_p^2(\partial/\partial t) + x_p\dot{x}_p\rho(\partial/\partial\rho) \qquad (7c)$$

and they obey the Lie algebra

$$[X_1, X_2] = -2X_3, \qquad [X_1, X_3] = -2X_2, \qquad [X_2, X_3] = 2X_1.$$

The latter is a three-dimensional subalgebra of the eight-dimensional algebra SL(3, R) (Leach 1980).

From our earlier work we know that the above symmetry operators are also Noether symmetry operators (Ray 1980). We may therefore use Noether's theorem to derive the constants of the motion associated with the above group operators. These constants of the motion have the form:

$$I_1 = \tfrac{1}{2}[x_p\dot{\rho} - \rho(\dot{x}_p + x_p^{-1} \cot 2\tau)]^2 \sin 2\tau + \tfrac{1}{2}(x_p/\rho)^2 \sin 2\tau + \tfrac{1}{2}(\rho/x_p)^2 \operatorname{cosec} 2\tau,$$

$$I_2 = \tfrac{1}{2}[x_p\dot{\rho} - \rho(\dot{x}_p - x_p^{-1} \tan 2\tau)]^2 \cos 2\tau + \tfrac{1}{2}(x_p/\rho)^2 \cos 2\tau + \tfrac{1}{2}(\rho/x_p)^2 \sec 2\tau,$$

$$I_3 = \tfrac{1}{2}(x_p\dot{\rho} - \rho\dot{x}_p)^2 + \tfrac{1}{2}(x_p/\rho)^2 + \tfrac{1}{2}(\rho/x_p)^2.$$

The latter invariant, which is independent of τ, has been widely used to study solutions to the nonlinear equation (1) both classically and quantum mechanically (Ray 1982).

To contrast these results with the linear oscillator we note that the equation

$$\ddot{\rho} + \omega^2(t)\rho = 0$$

possesses an eight-parameter Lie symmetry group SL(3, R) (Leach 1980) and a five-parameter subgroup of Noether symmetries (Lutzky 1978). For the Pinney equation (2) we have a three-parameter Lie and Noether symmetry group which is a subgroup of SL(3, R).

As a final point we mention that the symmetries of the coupled equations studied in Reid and Ray (1982) are associated with a one-parameter Lie symmetry group. For example, the equation

$$\ddot{\rho} + \omega^2(t)\rho = F(\rho/x)/x^3 \qquad (8)$$

possesses the Lie symmetry operator

$$X = x^2(\partial/\partial t) + x\dot{x}\rho(\partial/\partial\rho) \qquad (9)$$

where $x^2 = a$ and a is any solution to (4). For a given solution to (4) both the operator X in (9) and the differential equation (8) are determined and there is only a one-parameter Lie group of symmetries. For the Pinney equation we have $F = x^3/\rho^3$ and x does not appear in the differential equation (8), which then has the form (2). In this case the three integration constants of (4) give the three independent group operators (7) which we have discussed above. It is interesting, however, that we can still generate a three-parameter Lie group if we allow the differential equation (8) to vary as the parameters c_1, c_2, c_3 in (6) vary. In this case we generate the same three-parameter group as before but now this group is not a Lie symmetry group of a single differential equation but is associated with a set of differential equations that contain the parameters c_1, c_2, c_3. In this way we generate a group associated with a set of differential equations. As the parameters c_1, c_2, c_3 vary we move to a different differential equation and to a different operator. For fixed values of c_1, c_2, c_3, operator (9) generates a one-parameter Lie symmetry group for the corresponding differential equation of the set. Whether groups generated in this manner are useful in physics or not is an open question.

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